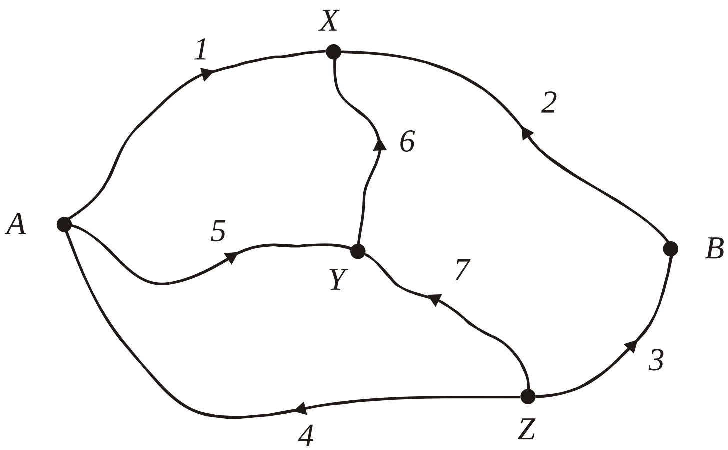
FREE NET LEVEL ADJUSTMENT

An example of a FREE NET LEVEL ADJUSTMENT is shown below. There are no fixed points (points of known height) in the network and hence there is a datum defect since the datum for heights is not defined. This means that there will be rank deficiencies (of one) in the design matrix **B** of the observation equations and the coefficient matrix **N** of the normal equations. This datum deficiency will be overcome by the addition of a single INNER CONSTRAINT EQUATION that enforces the condition that the centroid of the network remains unchanged. Since there are no fixed heights this constraint equation means that the sum of the adjusted heights is equal to zero.

The diagram below shows a level network of height differences observed between points *A, B, X, Y* and *Z*. The arrows on the diagram indicate the direction of rise. The table of height differences shows the height difference for each line of the network and the distance (in kilometres) of each level run.

Line	Height Diff	Dist (km)
1	6.345	1.6
2	4.235	2.5
3	3.060	1.0
4	0.920	4.0
5	3.895	1.6
6	2.410	1.25
7	4.820	2.0



The weight of each observed height difference is inversely proportional to the distance in kilometres and the adjustment model is: $P + \Delta H_{PQ} + v_{PQ} = Q$ where P and Q are the heights of points, ΔH is the height difference and v is the measurement residual. All points are considered to be "floating"; hence there will be a datum deficiency and no solution by the usual means.

Consider a set of n observation equations in u unknowns $(n > u)$ in the matrix form

$$\mathbf{v} + \mathbf{Bx} = \mathbf{f} \tag{1}$$

$\mathbf{v}$ is the $n \times 1$ vector of residuals

$\mathbf{x}$ is the $u \times 1$ vector of unknowns (or parameters)

$\mathbf{B}$ is the $n \times u$ matrix of coefficients (design matrix)

$\mathbf{f}$ is the $n \times 1$ vector of numeric terms (constants)

n is the number of equations (or observations)

u is the number of unknowns

$$\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \\ v_6 \\ v_7 \end{bmatrix} + \begin{bmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} A \\ B \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -l_1 \\ -l_2 \\ -l_3 \\ -l_4 \\ -l_5 \\ -l_6 \\ -l_7 \end{bmatrix} \tag{2}$$

$\mathbf{x}$ contains the heights, denoted by A, B, …and $\mathbf{f}$ contains the observed height differences denoted by $l_1, l_2, \ldots$

Note here that $\mathbf{B}$ is rank deficient. The rank of $\mathbf{B}$ is $rank(\mathbf{B}) \le u$ (and u = 5), and if we carry out elementary row transformations on the elements of $\mathbf{B}$ we obtain a reduced matrix of the form

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and can identify that the largest non-zero determinant is of order 4. Hence $rank(\mathbf{B}) = 4$ which is one less than its possible value. Hence $\mathbf{B}$ is rank deficient. The elements of the reduced matrix $\mathbf{B}$ above are obtained using the MATLAB function `rref()` which reduces a matrix to row echelon form (rref = reduced row echelon form) using Gauss Jordan elimination with partial pivoting.

Each observation (a height difference) has an associated weight (a measure of precision or variance) and the weight matrix **W** is a (square) matrix of order $n \times n$. Since the height differences are independent **W** is a diagonal matrix where each diagonal element $w_k = \dfrac{1}{\text{dist}_k}$ where the distance is in kilometres.

$$\mathbf{W} = \begin{bmatrix} w_1 & & & & \\ & w_2 & & & \\ & & w_3 & & \\ & & & \ddots & \\ & & & & w_7 \end{bmatrix}$$

The normal equations are

$$\mathbf{Nx} = \mathbf{t} \tag{3}$$

where

$$\mathbf{N} = \mathbf{B}^T\mathbf{WB} \quad \text{and} \quad \mathbf{t} = \mathbf{B}^T\mathbf{Wf} \tag{4}$$

N is the $u \times u$ coefficient matrix of the normal equations
t is the $u \times 1$ vector of numeric terms

The coefficient matrix **N** of the normal equations is

$$\mathbf{N} = \mathbf{B}^T\mathbf{WB} = \begin{bmatrix} w_1 + w_2 + w_3 & 0 & -w_1 & -w_5 & -w_4 \\ 0 & w_2 + w_3 & -w_2 & 0 & -w_3 \\ -w_1 & -w_2 & w_1 + w_2 + w_6 & -w_6 & 0 \\ -w_5 & 0 & -w_6 & w_5 + w_6 + w_7 & -w_7 \\ -w_4 & -w_3 & 0 & -w_7 & w_3 + w_4 + w_7 \end{bmatrix} \tag{5}$$

We can see here that **N** is singular since the last row is actually the sum of the previous four rows multiplied by minus one, i.e., $row_5 = \left(row_1 + row_2 + row_3 + row_4\right) \times -1$. Or one row is a linear combination of the others and the matrix is rank deficient. Since **N** is rank deficient and therefore singular (the determinant $|\mathbf{N}| = 0$) the inverse $\mathbf{N}^{-1}$ does not exist and there can be no solution for the heights by normal means.

Substituting values for the weights gives

$$\mathbf{N}=\begin{bmatrix} 1.500000 & 0.000000 & -0.625000 & -0.625000 & -0.250000 \\ 0.000000 & 1.400000 & -0.400000 & 0.000000 & -1.000000 \\ -0.625000 & -0.400000 & 1.825000 & -0.800000 & 0.000000 \\ -0.625000 & 0.000000 & -0.800000 & 1.925000 & -0.500000 \\ -0.250000 & -1.000000 & 0.000000 & -0.500000 & 1.750000 \end{bmatrix} \tag{6}$$

The vector of numeric terms is

$$\mathbf{t}=\begin{bmatrix} -w_1l_1+w_4l_4-w_5l_5 \\ -w_2l_2+w_3l_3 \\ w_1l_1+w_2l_2+w_6l_6 \\ w_5l_5-w_6l_6+w_7l_7 \\ -w_3l_3-w_4l_4-w_7l_7 \end{bmatrix}=\begin{bmatrix} -6.170000 \\ 1.366000 \\ 7.587625 \\ 2.916375 \\ -5.700000 \end{bmatrix} \tag{7}$$

Since $\mathbf{N}$ is <u>singular</u>, i.e., $|\mathbf{N}|=\mathbf{0}$ and $\mathbf{N}^{-1}$ does not exist; there is <u>no solution</u> by conventional means so a constraint equation will be added. The general form of constraint equations is

$$\mathbf{Cx}=\mathbf{g} \tag{8}$$

$\mathbf{C}$ is the $c\times u$ matrix of coefficients
$\mathbf{g}$ is the $c\times 1$ vector of numeric terms (constants)
c is the number of constraint equations

Combining the constraint equations (8) with the observation equations (1) and enforcing the least squares condition

$$\varphi=\mathbf{v}^T\mathbf{Wv}-2\mathbf{k}^T\left(\mathbf{Cx}-\mathbf{g}\right) \Rightarrow \text{minimum} \tag{9}$$

where $\mathbf{k}$ is the $c\times 1$ vector of *Lagrange multipliers*, gives rise to the set of equations

$$\left[\begin{array}{c:c} -\mathbf{N} & \mathbf{C}^T \\ \hdashline \mathbf{C} & \mathbf{0} \end{array}\right]\left[\begin{array}{c} \mathbf{x} \\ \hdashline \mathbf{k} \end{array}\right]=\left[\begin{array}{c} -\mathbf{t} \\ \hdashline \mathbf{g} \end{array}\right] \tag{10}$$

Equation (10) is the matrix equation for a constrained least squares solution of a survey network, where $\mathbf{C}$ is the coefficient matrix of the constraint equations $\mathbf{Cx}=\mathbf{g}$. In the case of INNER CONSTRAINTS and FREE NET ADJUSTMENTS, $\mathbf{Cx}=\mathbf{0}$ and equation (10) becomes

$$\begin{bmatrix} -\mathbf{N} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} -\mathbf{t} \\ \mathbf{0} \end{bmatrix} \tag{11}$$

Substituting the numeric terms from above gives

$$\left[\begin{array}{rrrrr|r} -1.500 & 0.000 & 0.625 & 0.625 & 0.250 & 1 \\ 0.000 & -1.400 & 0.400 & 0.000 & 1.000 & 1 \\ 0.625 & 0.400 & -1.825 & 0.800 & 0.000 & 1 \\ 0.625 & 0.000 & 0.800 & -1.925 & 0.500 & 1 \\ 0.250 & 1.000 & 0.000 & 0.500 & -1.750 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 0 \end{array}\right] \left[\begin{array}{c} A \\ B \\ X \\ Y \\ Z \\ \hline k_1 \end{array}\right] = \left[\begin{array}{r} 6.170000 \\ -1.366000 \\ -7.587625 \\ -2.916375 \\ 5.700000 \\ \hline 0.000000 \end{array}\right]$$

Noting here that the inner constraint equation is:

$$A + B + X + Y + Z = 0$$

The solution of this system of equations is

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = \begin{bmatrix} -\mathbf{N} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} \begin{bmatrix} -\mathbf{t} \\ \mathbf{g} \end{bmatrix} \tag{12}$$

The vector $\mathbf{x}$ contains the adjusted heights which are

$$\mathbf{x} = \begin{bmatrix} A \\ B \\ X \\ Y \\ Z \end{bmatrix} = \begin{bmatrix} -2.287824 \\ -0.164289 \\ 4.047945 \\ 1.619351 \\ -3.215183 \end{bmatrix}$$

Note here that the sum of the components of $\mathbf{x}$ is zero, which is also the height of the centroid, where the centroid of a level network is the average height. The norm of $\mathbf{x}$ denoted $\|\mathbf{x}\|$ is the square root of the sum of the squares and can be interpreted as the "length" of $\mathbf{x}$. From above $\|\mathbf{x}\| = 5.8827$. Free net adjustments produce minimum norm solutions, i.e., $\|\mathbf{x}\|$ is the smallest possible value. If A is held fixed at zero then the network can be solved by conventional means and the resulting vector $\mathbf{x}$ (in this case of length 4) has $\|\mathbf{x}\| = 7.7960$. Solutions to the level net could also be found by holding any other point fixed at zero. The set of norms for the sequence of solutions $A = 0$, $B = 0$, $X = 0$, $Y = 0$, $Z = 0$ is $\{7.7960 \quad 5.8942 \quad 10.7952 \quad 6.9078 \quad 9.2894\}$. We can see here that the free net solution (inner constraint) yields the minimum norm.

Rearranging equation (1) into $\mathbf{v} = \mathbf{f} - \mathbf{Bx}$ and substituting $\mathbf{x}$ gives the residuals $\mathbf{v}$

$$\mathbf{v}^T = \begin{bmatrix} -0.0092 & -0.0228 & -0.0091 & 0.0074 & 0.0122 & 0.0186 & 0.0145 \end{bmatrix}$$

The cofactor matrix of the adjusted heights is

$$\mathbf{Q}_{xx} = -\boldsymbol{\alpha} \tag{13}$$

where, from equation (12), the $u \times u$ matrix $\boldsymbol{\alpha}$ is obtained from the inverse

$$\begin{bmatrix} -\mathbf{N} & \mathbf{C}^T \\ \mathbf{C} & \mathbf{0} \end{bmatrix}^{-1} = \begin{bmatrix} \boldsymbol{\alpha} & \boldsymbol{\beta}^T \\ \boldsymbol{\beta} & \boldsymbol{\gamma} \end{bmatrix} \tag{14}$$

$$\mathrm{Q}_{\mathrm{xx}} = \begin{bmatrix} 0.499429 & -0.292579 & -0.001165 & 0.003450 & -0.209139 \\ -0.292579 & 0.603473 & -0.184850 & -0.244851 & 0.118802 \\ -0.001165 & -0.184850 & 0.399642 & 0.005019 & -0.218647 \\ 0.003450 & -0.244851 & 0.005019 & 0.381180 & -0.144799 \\ -0.209139 & 0.118802 & -0.218647 & -0.144799 & 0.453782 \end{bmatrix}$$

The elements of $\mathbf{Q}_{xx}$ are the estimates of variances of the adjusted heights. The trace of $\mathbf{Q}_{xx}$ (the sum of the diagonal elements) is $tr\left(\mathbf{Q}_{xx}\right) = 2.3375$. This value is the smallest sum of variances of adjusted quantities that could be obtained for this level network. If A is held fixed (at any value) then the network can be solved by conventional means and the resulting matrix $\mathbf{Q}_{xx}$ (in this case of order 4) will have $tr\left(\mathbf{Q}_{xx}\right) = 4.8347$. Free net adjustments (using inner constraints) produce minimum trace solutions or minimum variance solutions.

If we assign a height of 100 to A we have

$$A = 100.0000,\ \ B = 102.1235,\ \ X = 106.3358,\ \ Y = 103.9072,\ \ Z = 99.0726$$